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NOTE ON THE THEORY OF ROCKET MOTORS

By K. O. Friedrichs

As a basis for further discussion we report and comment on certain results in a paper by F. Malina concerning the theory of rocket motors*. Schematically a rocket motor consists of the combustion chamber, in which a gas in a state of high pressure is created, and the exhaust nozzle, through which the gas flows out into the atmosphere. Presently we are only concerned with the question of the efficiency of the exhaust nozzle. The main problem then is to determine the propulsing thrust resulting from the gas flow for nozzles of various shapes, the state of the gas in the combustion chamber being given.

A variety of simplifying assumptions are made: (i) the gas is polytrophic with an exponent $\gamma = 1.2$, say; (ii) the change of state of the streaming gas is adiabatic, losses through friction at the walls are ignored; (iii) the floor through the nozzle is steady; (iv) the nozzle diverges so gently that the classical "hydraulic" treatment may be applied.

A main result on the basis of these assumptions is that the propulsive thrust is a maximum when the floor is so guided that it has just expanded to atmospheric pressure at the end

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* Frank J. Malina, "Characteristics of the Rocket Motor Unit Based on the Theory of Perfect Gases."
Journal of the Franklin Institute, October, 1940

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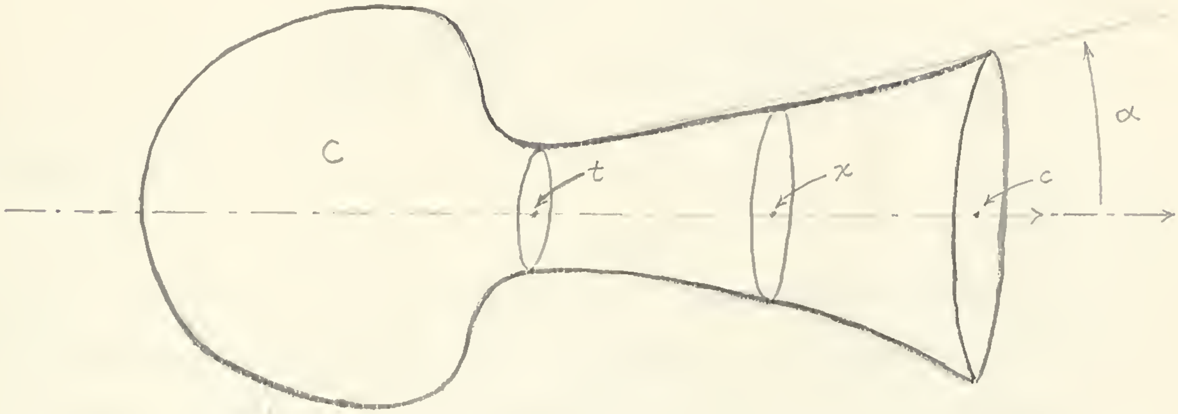
cross-section. If the gas is expanded below atmospheric pressure the propulsive thrust decreases very rapidly.

The hydraulic treatment of the nozzle flow may shortly be described: The points on the axis of the nozzle may be



characterized by their distance x from the chamber (c). At each cross-section of area f_x, the flow is defined by average pressure p_x, density ρ_x, and velocity v_x. The flow equations then are:

$$\rho_x v_x f_x = m; \tag{1}$$

the mass flux per unit time, is independent of x; so is the Bernoulli constant

$$\frac{\gamma}{\gamma - 1}\,\frac{p_x}{\rho_x} + \tfrac{1}{2}v_x^2 = \frac{\gamma}{\gamma - 1}\,\frac{p_c}{\rho_c}\,. \tag{2}$$

These two equations together with the adiabatic law

$$\frac{p_x}{(\rho_x)^{\gamma}} = \frac{p_c}{(\rho_c)^{\gamma}} \tag{3}$$

determine three of these quantities as functions of the fourth.

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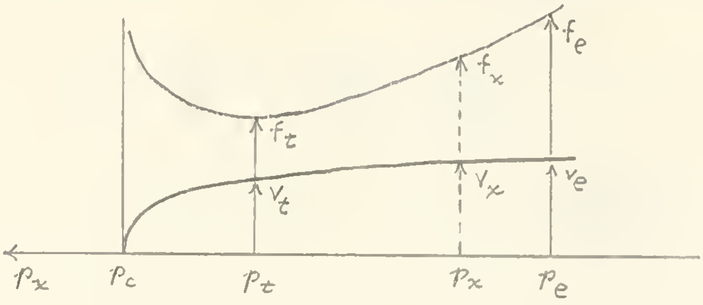
When the pressure p_x is taken as an independnet variable, the graphs of v_x and f_x are as in the diagram. The place where the cross-section area reaches its minimum indicates the "throat" (t).



Let F_x denote the thrust (in positive direction opposite to the flow) resulting from the gas pressure inside and the atmospheric pressure p_0 outside exerted against the chamber and the nozzle up to the cross-section (x). Then F_x is given by

$$(4) \qquad F_x = (p_x - p_0)\, f_x + mv_x \; ;$$

i.e. the thrust consists of a combustion due to pressure difference and one due to momentum transport. (Malina refines this formula by taking into account that the flow velocity will be radial rather than axial at the exit; he shows then that this correction is so small even up to nozzle angles up to 35° that it may be ignored.) The thrust F_x can easily be expressed as a function of the pressure p_x or the cross-section f_x. Malina gives the result in dimensionless form; the following formulation deviates slightly from his. We introduce the quantities

$$X = \frac{p_x}{p_t}$$

$$Y = \frac{v_x}{v_t} = \sqrt{\frac{\gamma + 1}{\gamma - 1}}\sqrt{1 - \left(\frac{X}{X_c}\right)^{\frac{\gamma - 1}{\gamma}}} \;;$$

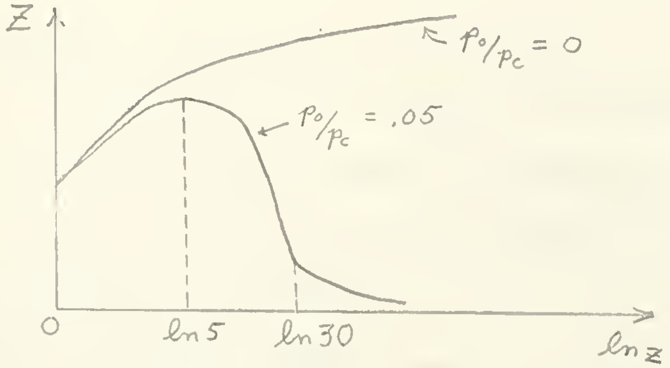
then

$$z = \frac{f_x}{f_t} = -\gamma\frac{dY}{dX} = \frac{1}{X^{\frac{1}{\gamma}}\,Y} \;,$$

and

$$Z = \frac{F_x}{F_t} = \left[Y - (X - X_0)\,\frac{dY}{dX}\right] \Big/ \left[\gamma + 1 - X_0\right] .$$

Malina shows the graphs of the thrust ratio Z as a function of the cross-section ratio z beyond the throat for various valuees of $\frac{p_0}{p_c}$. Two of these curves are indicated, one corresponding to vanishing outside pressure, the other to a small pressure ratio $\frac{p_0}{p_c} = .05$. One observes that the latter graph has a maximum and that beyond the maximum Z drops sharply and is sensitive to changes in $\frac{p_0}{p_c}$.



The maximum of Z occurs just when $X = X_0$, i.e. $p_c = p_0$, as follows e.g. from

$$dF_x = (p_x - p_0)df_x + f_x dp_x + m dv_x = (p_x - p_0)df_x \;.$$

This indicates that the maximum thrust occurs when the gas at the end cross-section has just expanded to atmospheric pressure;

[redacted]

the thrust is then solely due to momentum transport. (Malina does not mention this fact explicitly.)

When the end cross-section is so large that the exit pressure would be below outside pressure according to the theory as explained, the assumptions on which this theory is based are no longer valid; a different flow pattern involving shocks will occur. Malina determines the thrust under the assumption of a straight shock; the resulting change accounts for the break in the indicated graph.

Malina considers as an example a case with $\frac{p_c}{p_0} = 100$, $T = 11{,}100^\circ F$. $R = .0658$ B.T.U./lb./°F., $\gamma = 1.2$, and finds that the maximum thrust is obtained when $\frac{f_e}{f_t} = 11.85$, i.e. when the exit diameter is 3.44 times the throat diameter. The maximum thrust is given by

$$\frac{F}{p_0 f_e} = 13.6,$$

i.e. the thrust is 13.6 times the force which the atmosphere would exert against the end cross-section. Assuming a nozzle angle $\alpha = 15^\circ$, the length of the nozzle would be 9.1 times the throat radius or 2.64 times the radius of the end cross-section.

It seems conceivable that operation with much higher pressure ratios $\frac{p_c}{p_0}$ or higher temperatures is desired; in that case much longer nozzles would be required. This would reduce the danger

of over expansion and shocks within the nozzle* since actual nozzles then would rather be made too short. The hydraulic treatment then would be justified.

The design of a nozzle then should be based on the results of this treatment, such as explained in Stodola's book on turbines (cf. section 30), unless other physical circumstances are to be taken into account.

When a nozzle has been designed in the indicated way, it may be desirable to determine the flow through this nozzle more correctly in order to find out whether or not the hydraulic treatment was a sufficient approximation. This would require a mathematical analysis involving the solution of a partial differential equation, which would be rather tedious but possible. It seems hopelessly difficult, however, to obtain a sufficiently large manifold of such solutions, corresponding to various nozzles, which could serve as a basis for design.

K.O. Friedrichs
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* It is the question whether or not the jet outside of the nozzle is to be considered; in that case the ideas of v.Neumann about the flow pattern in a jet should be taken into account.

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